

On the invariant distributions of C^2 circle diffeomorphisms of irrational rotation number

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Although invariant measures are a fundamental tool in Dynamical Systems, very little is known about distributions (i.e. linear functionals defined on some space of smooth functions on the underlying space) that remain invariant under a dynamics. Perhaps the most general definite result in this direction is the remarkable theorem of A. Avila and A. Kocsard [1] according to which no C^∞ circle diffeomorphism of irrational rotation number has an invariant distribution different from (a scalar multiple of integration with respect to) the (unique) invariant (probability) measure. The main result of this Note is an analogous result in low regularity. Unlike [1] which involves very hard computations, our approach is more conceptual. It relies on the work of R. Douady and J.-C. Yoccoz concerning automorphic measures for circle diffeomorphisms [3].

Theorem A. *Circle diffeomorphisms of irrational rotation number that belong to the Denjoy class C^{1+bv} have no invariant 1-distributions different from the invariant measure.*

Here and in what follows, for $k \geq 0$, by a k -distribution we mean a (continuous) linear functional defined on the space of C^k real-valued functions on the circle. Notice that for all $k' > k$, a k -distribution may be seen as a k' -distribution, but the converse is false in general.

Theorem A allows to show an improved version of the Denjoy-Koksma inequality for C^1 test functions, thus extending Corollary C of [1] (valid for diffeomorphisms of class C^{11}) to C^{1+bv} diffeomorphisms. We omit the proof since it follows the very same lines of that of [1]. Indeed, the only new tool needed in [1] was the absence of invariant 1-distributions other than the invariant measure.

Corollary. *Let f be a C^{1+bv} circle diffeomorphism of irrational rotation number ρ . If (p_k/q_k) is the sequence of rational approximations of ρ , then for every C^1 function u on the circle, we have the uniform convergence*

$$u + u \circ f + u \circ f^2 + \dots + u \circ f^{q_k-1} - q_k \int_{S^1} u d\mu \longrightarrow 0,$$

where μ denotes the (unique) invariant probability measure of f .

Letting $u := \log(Df)$ whenever f is a C^2 diffeomorphism, this yields a well-known result of M. Herman: the sequence (f^{q_k}) converges to the identity in the C^1 topology (see [4, Chapitre VII]).

Finally, Theorem A is sharp in that diffeomorphisms with lower regularity may admit invariant 1-distributions.

Theorem B. *For each irrational angle ρ , there exists a C^1 circle diffeomorphism of rotation number ρ having an invariant 1-distribution different from the invariant measure. Moreover, such a diffeomorphism can be taken either being minimal or admitting a minimal invariant Cantor set.*

I. No invariant distributions for C^{1+bv} minimal diffeomorphisms. Given a C^{1+bv} circle diffeomorphism f of irrational rotation number, let μ be the invariant probability measure of f . In order to prove that f has no invariant 1-distribution other than μ , it suffices to show that for every C^1 function u on the circle of zero μ -mean, there exists a sequence of C^1 functions v_n such that the coboundaries

$$v_n \circ f - v_n$$

converge to u in the C^1 topology.¹ Indeed, if such a sequence exists, then for every invariant 1-distribution

¹Actually, a standard application of the Hahn-Banach theorem shows that this condition is also necessary, but we will not need this fact.

L ,

$$L(u) = \lim_{n \rightarrow \infty} [L(v_n \circ f) - L(v_n)] = 0.$$

Let us denote by λ the (normalized) Lebesgue measure on the circle. The existence of the desired sequence v_n is an almost direct consequence of the next Proposition.

Proposition. *There exists a sequence of continuous functions w_n such that*

$$(w_n \circ f)Df - w_n$$

uniformly converges to u' and for all n ,

$$\int_{S^1} w_n d\lambda = 0. \quad (1)$$

Assume this Proposition for a while. By integration from 0 to x , we obtain that the function $u(x) - u(0)$ is approximated by the sequence of continuous functions

$$x \mapsto \int_0^x (w_n \circ f)Df - \int_0^x w_n = \int_0^{f(x)} w_n - \int_0^x w_n - \int_0^{f(0)} w_n.$$

We let

$$v_n(x) := \int_0^x w_n d\lambda,$$

which is well defined due to (1). Then the function u is C^1 approximated by the sequence $v_n \circ f - v_n + c_n$, where c_n is a constant:

$$c_n := u(0) - \int_0^{f(0)} w_n.$$

By integration with respect to μ , one concludes that c_n necessarily converges to 0, thus yielding the desired approximating sequence and hence proving Theorem A.

For the proof of the Proposition above, the next particular case (corresponding to the displacement function $u(x) := f(x) - x$ minus its rotation number $\rho(f)$) will be crucial.

Lemma. *There exists a sequence of continuous functions \hat{w}_k such that*

$$(\hat{w}_k \circ f)Df - \hat{w}_k$$

uniformly converges to $Df(x) - 1$ and for all k ,

$$\int_{S^1} \hat{w}_k d\lambda = 0.$$

Proof. Let

$$\hat{w}_k := -\frac{1}{q_k} [1 + Df + Df^2 + \dots + Df^{q_k-1}] + 1,$$

where (q_k) is the sequence of denominators in the rational approximation of the rotation number of f . Notice that for all $k \geq 1$,

$$\int_{S^1} \hat{w}_k d\lambda = 0.$$

Moreover,

$$\begin{aligned} \hat{w}_k(f(x))Df(x) &= -\frac{Df(x)}{q_k} [1 + Df(f(x)) + Df^2(f(x)) + \dots + Df^{q_k-1}(f(x))] + Df(x) \\ &= -\frac{1}{q_k} [Df(x) + Df^2(x) + \dots + Df^{q_k}(x)] + Df(x) \\ &= \hat{w}_k(x) - 1 + \frac{1}{q_k} [1 - Df^{q_k}(x)] + Df(x), \end{aligned}$$

hence

$$Df(x) - 1 = (\hat{w}_k \circ f)Df - \hat{w}_k + \frac{1}{q_k}[Df^{q_k} - 1].$$

The desired convergence then follows from the Denjoy inequality:

$$|Df^{q_k}| \leq \exp(V),$$

where V denotes the total variation of the logarithm of Df (see [6, Chapter 3]). \square

Let us now come back to the Proposition. For the proof, let us recall that given $s \in \mathbb{R}$, an s -automorphic measure for f is a probability measure ν on the circle such that for every continuous function φ ,

$$\int_{S^1} \varphi d\nu = \int_{S^1} (\varphi \circ f)(Df)^s d\nu.$$

For a C^{1+bv} circle diffeomorphisms of irrational rotation number, such a measure exists and is unique for each s (see [3]). The (unique) 1-automorphic measure is hence the Lebesgue measure. Moreover, the uniqueness holds (up to a scalar factor) even in the context of signed finite measures (i.e. linear functionals defined on the space of continuous functions).

Now, since

$$\int_{S^1} u' d\lambda = 0,$$

a standard application of the Hahn-Banach theorem (to the functional $w \mapsto (w \circ f)Df - w$) yields a sequence of continuous functions \bar{w}_n such that

$$(\bar{w}_n \circ f)Df - \bar{w}_n$$

uniformly converges to u' . Indeed, assume otherwise. Then u' does not belong to the closure of the set of functions of the form $(w \circ f)Df - w$. The latter set being convex, there exists a linear functional L which is identically zero on this set but $L(u') = 1$. The former condition means that, as a signed measure, L is 1-automorphic. It is hence a nonzero multiple of the Lebesgue measure, which is absurd since $L(u') = 1$ and u' has zero mean with respect to λ .

The problem with the approximating sequence \bar{w}_n above is that it is unclear whether one can ensure that

$$c_n := \int_{S^1} \bar{w}_n d\lambda$$

equals zero. To solve this problem, we consider the functions $\tilde{w}_n := \bar{w}_n - c_n$, which obviously have zero mean (with respect to λ). Given $\varepsilon > 0$, we may choose n such that the absolute value of the difference between

$$(\tilde{w}_n \circ f)Df - \tilde{w}_n$$

and

$$u' - c_n(Df - 1)$$

is smaller than or equal to $\varepsilon/2$. Moreover, the Lemma yields $k = k_n$ such that

$$c_n |(Df - 1) - [(\hat{w}_k \circ f)Df - \hat{w}_k]| \leq \frac{\varepsilon}{2}.$$

Putting all of this together, we get that

$$|u' - [(\tilde{w}_n \circ f - c_n \hat{w}_k \circ f)Df - (\tilde{w}_n - c_n \hat{w}_k)]| \leq \varepsilon,$$

which together with

$$\int_{S^1} (\tilde{w}_n - c_n \hat{w}_k) d\lambda = 0$$

proves the Proposition.

II. Examples of C^1 diffeomorphisms with invariant distributions. We first deal with diffeomorphisms with a minimal invariant Cantor set. To do this, we next recall a (particular case of a) construction from [3, Section 5] and then show how this provides an example of a C^1 circle diffeomorphism of irrational rotation number admitting invariant 1-distributions different from the invariant measure.

Theorem [R. Douady & J.-C. Yoccoz]. *For each irrational angle ρ , there exists a C^1 circle diffeomorphism f which is a Denjoy counter-example and satisfies the following property: for a certain point x_0 belonging to the complement of the exceptional minimal Cantor set, one has*

$$S := \sum_{n \in \mathbb{Z}} Df^n(x_0) < \infty.$$

In particular, the (probability) measure

$$\nu := \frac{1}{S} \sum_{n \in \mathbb{Z}} Df^n(x_0) \delta_{f^n(x_0)}$$

is 1-automorphic.

Let us fix f as above, and let μ be its (unique) invariant measure. Let us consider the linear functional (compare [2])

$$L: u \mapsto \int_{S^1} u' d\nu,$$

defined on the space of C^1 functions u on the circle. Then L is f -invariant. Indeed,

$$L(u \circ f) = \int_{S^1} (u \circ f)' d\nu = \int_{S^1} (u' \circ f) Df d\nu = \int_{S^1} u' d\nu = L(u),$$

where the third equality follows from that ν is 1-automorphic.

To see that L is different from (a multiple of the integration with respect to the) invariant measure μ , we let I be the connected component of the complement of the exceptional minimal set K that contains x_0 . Then K coincides with the support of μ , so that for every function u supported on I , we have

$$\int_{S^1} u d\mu = 0.$$

For such a function, one has

$$L(u) = \int_{S^1} u' d\nu = \frac{u'(x_0)}{S}.$$

However, this expression can take any real value for different functions u as above.

The construction of C^1 minimal diffeomorphisms with invariant distributions follows a similar strategy. The main tool is a recent result from [5] which in a certain sense can be considered as a measurable counterpart of that of Douady and Yoccoz.

Theorem [H. Kodama & S. Matsumoto]. *For each irrational rotation angle ρ there exists a minimal C^1 circle diffeomorphism f of rotation number ρ admitting a measurable fundamental domain C (that is, a measurable set that is disjoint from all of its iterates). Moreover, for each point $x_0 \in C$, one has*

$$S := \sum_{n \in \mathbb{Z}} Df^n(x_0) < \infty.$$

As before, the last condition implies that the measure

$$\nu := \frac{1}{S} \sum_{n \in \mathbb{Z}} Df^n(x_0) \delta_{f^n(x_0)}$$

is 1-automorphic for f , so that we may again define the f -invariant 1-distribution

$$L: u \mapsto \int_{S^1} u' d\nu$$

for C^1 functions u on the circle. However, unlike the previous case, showing that L is nontrivial on the kernel of μ is not geometrically obvious. Nevertheless, there is a simple general argument that shows this fact (which also applies to the previous case), as we next explain.

By simple integration, every continuous function on the circle of zero λ -mean can be seen as the derivative of a C^1 function having zero μ -mean. As a consequence, if L were trivial on the kernel of μ , then for every continuous function v on the circle, we would have

$$0 = \int_{S^1} \left(v - \int_{S^1} v d\lambda \right) d\nu = \int_{S^1} v d\nu - \int_{S^1} v d\lambda.$$

This would mean that $\nu = \lambda$, which is absurd.

Remark. Although Douady-Yoccoz' examples can not be made $C^{1+\tau}$ for any $\tau > 0$, the Denjoy counter-examples extensively described in [6] carry 1-automorphic atomic measures in the same way as those of [3]. The case of minimal diffeomorphisms is more complicated. Indeed, Kodama-Matsumoto's examples are neither $C^{1+\tau}$, and it is unclear whether one can produce 1-automorphic measures by following a similar construction. (Actually, there is no known minimal $C^{1+\tau}$ circle diffeomorphism which is non ergodic with respect to the Lebesgue measure.) This raises the question of the existence of minimal $C^{1+\tau}$ circle diffeomorphisms of irrational rotation number having invariant 1-distributions other than the invariant measure.

Question. Does there exist a C^2 circle diffeomorphism of irrational rotation number carrying a 2-invariant distribution different from the invariant measure ?

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